

Explicit Orbit Classification of Reducible Jordan Algebras and Freudenthal Triple Systems

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ABSTRACT

We determine explicit orbit representatives of *reducible* Jordan algebras and of their corresponding Freudenthal triple systems. This work has direct application to the classification of extremal black hole solutions of $\mathcal{N} = 2, 4$ locally supersymmetric theories of gravity coupled to an arbitrary number of Abelian vector multiplets in $D = 4, 5$ space-time dimensions.

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1 Introduction

The present investigation is devoted to the study of the explicit representatives of the orbits of *reducible* cubic Jordan algebras, and of their corresponding Freudenthal triple systems (FTS). This is in the spirit of previous analyses by Shukuzawa [1], which in turn was inspired *e.g.* by Jacobson [2] and Krutelevich [3–5]. By reducible we mean here that the cubic norm of the underlying Jordan algebra is a factorisable homogeneous polynomial of degree 3, as opposed to the irreducible, *i.e.* *non-factorisable*, cases treated in the previous works [1, 5]. In a companion paper [6], the results of the present analysis have been used to classify extremal black hole solutions in locally supersymmetric theories of gravity with $\mathcal{N} = 2$ or 4 supercharges in $D = 4$ and 5 space-time dimensions, coupled to an arbitrary number of (Abelian) vector multiplets. This paper aims at completing and refining previous investigations [7–15], and it also provides an alternative approach with respect to the analysis based on nilpotent orbits of symmetry groups characterising the $D = 3$ time-like reduced gravity theories [16–19].

The paper is organised as follows.

Sec. 2 deals with cubic Jordan algebras. After some introductory background in Secs. 2.1 and 2.2, the explicit representatives of the orbits of *magic* Jordan algebras [1–3] are recalled in Sec. 2.3.1. Original results for *reducible* Lorentzian spin factors (then generalised to an arbitrary pseudo-Euclidean signature) are derived in Sec. 2.3.2, considering both “large” (rank 4) and “small” orbits (these latter further split into three sub-classes, ranging from rank 3 to 1). The peculiar cases of the so-called $\mathcal{N} = 2$ *STU*, ST^2 and T^3 supergravity models in $D = 5$ are considered in Sec. 2.3.3.

Then, Sec. 3 studies the Freudenthal triple systems. These are defined both axiomatically and in relation to possibly underlying cubic Jordan algebras, respectively in Secs. 3.1 and 3.2, and their automorphism group is recalled in Sec. 3.3. The explicit orbit representatives for *magic* Freudenthal triple systems [1, 4] are considered in Sec. 3.4.1, and then original results on the orbit representatives of Freudenthal systems associated to *reducible* spin factors are derived in Sec. 3.4.2. As for the Jordan algebra analysis worked out in Sec. 2, the *STU*, ST^2 and T^3 models deserve a separate treatment, which is given in the concluding Sec. 3.4.3.

2 Orbits of Cubic Jordan Algebras

2.1 Construction

A Jordan algebra \mathfrak{J} is vector space defined over a ground field \mathbb{F} (not of characteristic 2) equipped with a bilinear product satisfying [20–22]

$$A \circ B = B \circ A, \quad A^2 \circ (A \circ B) = A \circ (A^2 \circ B), \quad \forall A, B \in \mathfrak{J}. \quad (2.1)$$

However, the 5-dimensional supergravities [23–25] are characterised specifically by the class of *cubic* Jordan algebras, developed in [26, 27]. We sketch their construction here, following the presentation of [28].

Definition 1 (Cubic norm). *A cubic norm is a homogeneous map of degree three*

$$N : V \rightarrow \mathbb{F}, \quad \text{s.t.} \quad N(\alpha A) = \alpha^3 N(A), \quad \forall \alpha \in \mathbb{F}, A \in V \quad (2.2)$$

such that its linearization,

$$N(A, B, C) := \frac{1}{6}(N(A + B + C) - N(A + B) - N(A + C) - N(B + C) + N(A) + N(B) + N(C)) \quad (2.3)$$

is trilinear.

Let V be a vector space equipped with a cubic norm. If V further contains a base point $N(c) = 1, c \in V$ one may define the following four maps:

1. The trace,

$$\text{Tr}(A) = 3N(c, c, A), \quad (2.4a)$$

2. A quadratic map,

$$S(A) = 3N(A, A, c), \quad (2.4b)$$

3. A bilinear map,

$$S(A, B) = 6N(A, B, c), \quad (2.4c)$$

4. A trace bilinear form,

$$\text{Tr}(A, B) = \text{Tr}(A) \text{Tr}(B) - S(A, B). \quad (2.4d)$$

A cubic Jordan algebra \mathfrak{J} with multiplicative identity $\mathbb{1} = c$ may be derived from any such vector space if N is *Jordan cubic*.

Definition 2 (Jordan cubic norm). *A cubic norm is Jordan if*

1. *The trace bilinear form (2.4d) is non-degenerate.*
2. *The quadratic adjoint map, $\sharp : \mathfrak{J} \rightarrow \mathfrak{J}$, uniquely defined by $\text{Tr}(A^\sharp, B) = 3N(A, A, B)$, satisfies*

$$(A^\sharp)^\sharp = N(A)A, \quad \forall A \in \mathfrak{J}. \quad (2.5)$$

The Jordan product is given by

$$A \circ B := \frac{1}{2}(A \times B + \text{Tr}(A)B + \text{Tr}(B)A - S(A, B)\mathbb{1}), \quad (2.6)$$

where,

$$A \times B := (A + B)^\sharp - A^\sharp - B^\sharp. \quad (2.7)$$

Finally, the Jordan triple product is defined as

$$\{A, B, C\} := (A \circ B) \circ C + A \circ (B \circ C) - (A \circ C) \circ B. \quad (2.8)$$

Definition 3 (Irreducible idempotent). *An element $E \in \mathfrak{J}$ is an irreducible idempotent if*

$$E \circ E = E, \quad \text{Tr}(E) = 1. \quad (2.9)$$

2.2 Symmetries

There are many good references on the symmetries associated with Jordan algebras and, in particular, on the exceptional Lie groups. Here we have used [29–32] and in particular [1, 5, 33]. In the following we restrict our attention to the case $\mathbb{F} = \mathbb{R}$.

Definition 4 (Automorphism group $\text{Aut}(\mathfrak{J})$). *Invertible \mathbb{R} -linear transformations τ preserving the Jordan product:*

$$\begin{aligned}\text{Aut}(\mathfrak{J}) &:= \{\tau \in \text{Iso}_{\mathbb{R}}(\mathfrak{J}) \mid \tau(A \circ B) = \tau A \circ \tau B\} \\ &= \{\tau \in \text{Iso}_{\mathbb{R}}(\mathfrak{J}) \mid \tau(A \times B) = \tau A \times \tau B\} \\ &= \{\tau \in \text{Iso}_{\mathbb{R}}(\mathfrak{J}) \mid N(\tau A) = N(A), \tau \mathbf{1} = \mathbf{1}\}.\end{aligned}\tag{2.10}$$

The equivalence of these various definitions may be found *e.g.* in [33]. The corresponding Lie algebra is given by the set of *derivations*,

$$\begin{aligned}\mathfrak{Aut}(\mathfrak{J}) \sim \mathfrak{der}(\mathfrak{J}) &= \{\delta \in \text{Hom}_{\mathbb{R}}(\mathfrak{J}) \mid \delta(A \circ B) = \delta A \circ B + A \circ \delta B\} \\ &= \{\delta \in \text{Hom}_{\mathbb{R}}(\mathfrak{J}) \mid \delta A \times B + A \times \delta B = 0\} \\ &= \{\delta \in \text{Hom}_{\mathbb{R}}(\mathfrak{J}) \mid N(\delta A, A, A) = 0, \delta \mathbf{1} = 0\}.\end{aligned}\tag{2.11}$$

Definition 5 (Reduced structure group $\text{Str}_0(\mathfrak{J})$). *Invertible \mathbb{R} -linear transformations τ preserving the cubic norm:*

$$\begin{aligned}\text{Str}_0(\mathfrak{J}) &:= \{\tau \in \text{Iso}_{\mathbb{R}}(\mathfrak{J}) \mid N(\tau A) = N(A)\} \\ &= \{\tau \in \text{Iso}_{\mathbb{R}}(\mathfrak{J}) \mid {}^t\tau^{-1}(A \times B) = \tau A \times \tau B\}.\end{aligned}\tag{2.12}$$

The proof of the equivalence of such two definitions may be found *e.g.* in [33]. The corresponding Lie algebra $\mathfrak{Str}_0(\mathfrak{J})$ is given by

$$\begin{aligned}\mathfrak{Str}_0(\mathfrak{J}) &= \{\phi \in \text{Hom}_{\mathbb{R}}(\mathfrak{J}) \mid N(\phi A, A, A) = 0\} \\ &= \{\phi \in \text{Hom}_{\mathbb{R}}(\mathfrak{J}) \mid {}^t\phi(A \times B) = \phi A \times B + A \times \phi B\}.\end{aligned}\tag{2.13}$$

The reduced structure algebra may be decomposed with respect to the automorphism algebra, as follows:

$$\mathfrak{Str}_0(\mathfrak{J}) = L'(\mathfrak{J}) \oplus \mathfrak{der}(\mathfrak{J}),\tag{2.14}$$

where $L'(\mathfrak{J})$ denotes the set of left Jordan products by traceless elements, $L_A(B) = A \circ B$ where $\text{Tr}(A) = 0$.

Definition 6 (Structure group $\text{Str}(\mathfrak{J})$). *Invertible \mathbb{R} -linear transformations τ preserving the cubic norm up to a fixed scalar factor,*

$$\text{Str}(\mathfrak{J}) := \{\tau \in \text{Iso}_{\mathbb{R}}(\mathfrak{J}) \mid N(\tau A) = \lambda N(A), \lambda \in \mathbb{R}\}.\tag{2.15}$$

The corresponding Lie algebra $\mathfrak{Str}(\mathfrak{J})$ is given by,

$$\mathfrak{Str}(\mathfrak{J}) = L(\mathfrak{J}) \oplus \mathfrak{der}(\mathfrak{J}),\tag{2.16}$$

where $L(\mathfrak{J})$ denotes the set of left Jordan products $L_A(B) = A \circ B$.

A cubic Jordan algebra element may be assigned a $\text{Str}(\mathfrak{J})$ invariant *rank* [2].

Definition 7 (Cubic Jordan algebra rank). *A non-zero element $A \in \mathfrak{J}$ has a rank given by:*

$$\begin{aligned}\text{Rank} A = 1 &\Leftrightarrow A^\sharp = 0; \\ \text{Rank} A = 2 &\Leftrightarrow N(A) = 0, A^\sharp \neq 0; \\ \text{Rank} A = 3 &\Leftrightarrow N(A) \neq 0.\end{aligned}\tag{2.17}$$

2.3 Explicit Orbit Representatives

2.3.1 Magic Jordan Algebras

We denote by $\mathfrak{J}_3^{\mathbb{A}}$ the cubic Jordan algebra of 3×3 Hermitian matrices with entries in a composition algebra \mathbb{A} [22]. An arbitrary element may be written as

$$A = \begin{pmatrix} \alpha & c & \bar{b} \\ \bar{c} & \beta & a \\ b & \bar{a} & \gamma \end{pmatrix}, \quad \text{where } \alpha, \beta, \gamma \in \mathbb{R} \quad \text{and} \quad a, b, c \in \mathbb{A}. \quad (2.18)$$

The cubic norm (2.2) is defined as,

$$N(A) := \alpha\beta\gamma - \alpha a \bar{a} - \beta b \bar{b} - \gamma c \bar{c} + (ab)c + \bar{c}(\bar{b}\bar{c}), \quad (2.19)$$

which for associative \mathbb{A} coincides with the matrix determinant.

The Jordan product (2.6) is given by

$$A \circ B := \frac{1}{2}(AB + BA), \quad A, B \in \mathfrak{J}_3^{\mathbb{A}}, \quad (2.20)$$

where juxtaposition denotes the conventional matrix product. Evidently, $c = \text{diag}(1, 1, 1)$ is a base point and the corresponding Jordan algebra maps are given by

$$\begin{aligned} \text{Tr}(A) &= \text{tr}(A), \\ \text{Tr}(A, B) &= \text{tr}(A \circ B), \end{aligned} \quad (2.21a)$$

where tr is the conventional matrix trace. The quadratic adjoint (2.5) is given by

$$A^\sharp = \begin{pmatrix} \beta\gamma - |a|^2 & \bar{b}\bar{a} - \gamma c & ca - \beta\bar{b} \\ ab - \gamma\bar{c} & \alpha\gamma - |b|^2 & \bar{c}\bar{b} - \alpha a \\ \bar{a}\bar{c} - \beta b & bz - \alpha\bar{a} & \beta\alpha - |c|^2 \end{pmatrix}. \quad (2.22)$$

The irreducible idempotents are given by

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.23)$$

The reduced structure group $\text{Str}_0(\mathfrak{J}_3^{\mathbb{A}})$ is given by $\text{SL}(3, \mathbb{R})$, $\text{SL}(3, \mathbb{C})$, $\text{SU}^*(6)$ and $E_{6(-26)}$ for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and \mathbb{O} , respectively. These are nothing but the U-duality groups of the so-called magic Maxwell-Einstein $\mathcal{N} = 2$ supergravity theories in $D = 5$ space-time dimensions [24, 25].

Theorem 8 (Shukuzawa, 2006 [1]). *Every element $A \in \mathfrak{J}_3^{\mathbb{O}}$ of a given rank is $\text{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ related to one of the following canonical forms:*

1. *Rank 1*

- (a) $A_{1a} = (1, 0, 0) = E_1$
- (b) $A_{1b} = (-1, 0, 0) = -E_1$

2. *Rank 2*

- (a) $A_{2a} = (1, 1, 0) = E_1 + E_2$
- (b) $A_{2b} = (-1, 1, 0) = -E_1 + E_2$

$$(c) \ A_{2c} = (-1, -1, 0) = -E_1 - E_2$$

3. *Rank 3*

$$(a) \ A_{3a} = (1, 1, k) = E_1 + E_2 + kE_3$$

$$(b) \ A_{3b} = (-1, -1, k) = -E_1 - E_2 + kE_3$$

The result clearly also holds for $\mathfrak{J}_3^{\mathbb{R}}, \mathfrak{J}_3^{\mathbb{C}}, \mathfrak{J}_3^{\mathbb{H}}$. It is worth remarking here that the analogue of this Theorem for $\mathfrak{J}_3^{\mathbb{O}^s}$ has been proved in [2] (see also [3]; this Theorem also holds for $\mathfrak{J}_3^{\mathbb{H}^s}$ and $\mathfrak{J}_3^{\mathbb{C}^s}$).

2.3.2 Lorentzian Spin Factors

Given a vector space V over a field \mathbb{F} with a non-degenerate quadratic form $Q(v), v \in V$, containing a base point $Q(c_0) = 1$, we may construct a cubic Jordan algebra $\mathfrak{J}_V = \mathbb{F} \oplus V$ with base point $c = (1; c_0) \in \mathfrak{J}_V$ and cubic norm,

$$N(A) = aQ(v), \quad (a; v) \in \mathfrak{J}_V. \quad (2.24)$$

See, for example, [5, 28]. In particular, the Lorentzian spin factors¹ $\mathfrak{J}_{1,n-1} := \mathbb{R} \oplus \Gamma_{1,n-1}$ are defined by the cubic norm,

$$N(A) = aa_\mu a^\mu = a(a_0^2 - a_i a_i), \quad \text{where } a \in \mathbb{R} \quad \text{and} \quad a_\mu \in \mathbb{R}^{1,n-1} \quad (2.25)$$

for elements $A \in \mathfrak{J}_{1,n-1}$,

$$A = (a; a_\mu) = (a; a_0, a_i). \quad (2.26)$$

For notational convenience, we will often only write the first three components $(a; a_0, a_1)$ if $a_i = 0$ for $i > 1$. The linearisation of the cubic norm is given by

$$N(A, B, C) = \frac{1}{3}(ab_\mu c^\mu + ca_\mu b^\mu + bc_\mu a^\mu). \quad (2.27)$$

Evidently, $E = (1; 1, 0)$ is a base point and the corresponding Jordan algebra maps are given by

$$\begin{aligned} \text{Tr}(A) &= a + 2a_0, \\ S(A) &= 2aa_0 + a_\mu a^\mu, \\ S(A, B) &= 2(ab_0 + ba_0 + a_\mu b^\mu), \\ \text{Tr}(A, B) &= ab + 2(a_0 b_0 + a_i b_i). \end{aligned} \quad (2.28)$$

Using $\text{Tr}(A^\#, B) = 3N(A, A, B)$ one obtains the quadratic adjoint

$$A^\# = (a_\mu a^\mu; aa^\mu), \quad (2.29)$$

where the index has been raised using the “most minus” Lorentzian metric $\eta^{\mu\nu} = \text{diag}(1, -1, -1, \dots, -1)$. Its linearisation $A \times B = (A + B)^\# - A^\# - B^\#$ yields

$$A \times B = (2a_\mu b^\mu; ba^\mu + ab^\mu). \quad (2.30)$$

It is not difficult to verify that

$$A^{\#\#} = N(A)A, \quad (2.31)$$

¹In general, $\Gamma_{m,n}$ is a Jordan algebra with a quadratic form of pseudo-Euclidean signature (m, n) , *i.e.* the Clifford algebra of $O(m, n)$ [22].

so that the quadratic adjoint is indeed Jordan cubic. Hence, we obtain a well defined Jordan algebra with Jordan product defined by Eq. (2.6), yielding

$$A \circ B = (ab; a_0b_0 + \sum_j a_jb_j, a_0b_i + b_0a_i). \quad (2.32)$$

Three irreducible idempotents are given by

$$E_1 = (1; 0), \quad E_2 = (0; \frac{1}{2}, \frac{1}{2}), \quad E_3 = (0; \frac{1}{2}, -\frac{1}{2}). \quad (2.33)$$

The reduced structure group $\text{Str}_0(\mathfrak{J}_{1,n-1})$ is given by $\text{SO}(1, 1) \times \text{SO}(1, n-1)$, where we have chosen to restrict to determinant 1 matrices. Explicitly, A transforms as

$$(a; a_\mu) \mapsto (e^{2\lambda}a; e^{-\lambda}\Lambda_\mu{}^\nu a_\nu), \quad \text{where } \lambda \in \mathbb{R}, \Lambda \in \text{SO}(1, n-1). \quad (2.34)$$

Theorem 9. *For $n \geq 2$ every element $A = (a; a_\mu) \in \mathfrak{J}_{1,n-1}$ of a given rank is $\text{SO}(1, 1) \times \text{SO}(1, n-1)$ related to one of the following canonical forms:*

1. Rank 1

- (a) $A_{1a} = (1; 0) = E_1$
- (b) $A_{1b} = (-1; 0) = -E_1$
- (c) $A_{1c} = (0; \frac{1}{2}, \frac{1}{2}) = E_2$

2. Rank 2

- (a) $A_{2a} = (0; 1, 0) = E_1 + E_2$
- (b) $A_{2b} = (0; 0, 1) = E_1 - E_2$
- (c) $A_{2c} = (1; \frac{1}{2}, \frac{1}{2}) = E_1 + E_2$
- (d) $A_{2d} = (-1; \frac{1}{2}, \frac{1}{2}) = -E_1 + E_2$

3. Rank 3

- (a) $A_{3a} = (1; \frac{1}{2}(1+k), \frac{1}{2}(1-k)) = E_1 + E_2 + kE_3$
- (b) $A_{3b} = (-1; \frac{1}{2}(1+k), \frac{1}{2}(1-k)) = -E_1 + E_2 + kE_3$

Note, if one restricts to the identity-connected component of $\text{SO}(1, n-1)$, each of the orbits A_{1c} , A_{2c} and A_{2d} splits into two cases, A_{1c}^\pm , A_{2c}^\pm and A_{2d}^\pm , corresponding to the future and past light cones. Similarly, A_{2a} splits into two disconnected components, A_{2a}^\pm , corresponding to the future and past hyperboloids. For $k > 0$ the orbits A_{3a} and A_{3b} also split into disconnected future and past hyperboloids, A_{3a}^\pm and A_{3b}^\pm .

Proof. $\text{Rank} A = 1 \Rightarrow$

$$A^\sharp = (a_\mu a^\mu, aa^\mu) = 0, \quad (a; a_\mu) \neq 0. \quad (2.35)$$

This corresponds to two cases: (i) $a_\mu = 0, a \neq 0$ or (ii) $a = 0, a_\mu a^\mu = 0, a_\mu \neq 0$. In case (i) we have

$$A = (a; 0) \mapsto (e^{2\lambda}a; 0) = (\pm 1; 0). \quad (2.36)$$

In case (ii) we have

$$A = (0; a_\mu) \mapsto (0; \Lambda_\mu{}^\nu a_\nu) = (0; \frac{1}{2}, \frac{1}{2}). \quad (2.37)$$

$\text{Rank} A = 2 \Rightarrow$

$$N(A) = aa_\mu a^\mu = 0, \quad A^\sharp = (a_\mu a^\mu; aa^\mu) \neq 0. \quad (2.38)$$

This corresponds to two cases: (i) $a = 0, a_\mu a^\mu \neq 0$ or (ii) $a_\mu a^\mu = 0, a \neq 0, a_\mu \neq 0$. In case (i) we have

$$A = (0; a_\mu) \mapsto (0; e^{-\lambda} \Lambda_\mu{}^\nu a_\nu) = (0; 1, 0) \quad \text{or} \quad (0; 0, 1). \quad (2.39)$$

In case (ii) we have

$$A = (a; a_\mu) \mapsto (e^{2\lambda} a; e^{-\lambda} \Lambda_\mu{}^\nu a_\nu) = (\pm 1; \frac{1}{2}, \frac{1}{2}). \quad (2.40)$$

$$\text{Rank } A = 3 \Rightarrow$$

$$N(A) = a a_\mu a^\mu \neq 0. \quad (2.41)$$

Hence,

$$A = (a; a_\mu) \mapsto (e^{2\lambda} a; e^{-\lambda} \Lambda_\mu{}^\nu a_\nu) = (\pm 1; \frac{1}{2}(1+k), \frac{1}{2}(1-k)). \quad (2.42)$$

where $N(A) = \pm k$. \square

Note, the Lorentzian spin-factor construction may be generalised to an arbitrary pseudo-Euclidean signature Jordan algebra $\mathfrak{J}_{p-1, q-1} = \mathbb{R} \oplus \Gamma_{p-1, q-1}$ by defining the cubic norm,

$$N(A) = a a_\mu a^\mu, \quad (a; a_\mu) \in \mathfrak{J}_{p-1, q-1}, \quad (2.43)$$

where the index has been raised with a $\{+^{p-1}, -^{q-1}\}$ signature pseudo-Euclidean metric. The same base point $c = (1; 1, 0)$ may be used. A more “democratic” choice valid for any $p \geq 2$ is given by

$$c = \frac{1}{\sqrt{p-1}} (\sqrt{p-1}; \underbrace{1, 1, \dots, 1}_{p-1}, \underbrace{0, 0, \dots, 0}_{q-1}),$$

although it obscures some of the symmetries by unnecessarily complicating the basic identities. In this case the reduced structure group is given by,

$$\text{Str}_0(\mathfrak{J}_{p-1, q-1}) = \text{SO}(1, 1) \times \text{SO}(p-1, q-1). \quad (2.44)$$

The analysis goes through analogously to the Lorentzian case so we will not treat it in detail here. See, for example, [6, 34] for further details.

2.3.3 Special Cases: $\mathfrak{J}_{3\mathbb{R}}, \mathfrak{J}_{2\mathbb{R}}$ and $\mathfrak{J}_{\mathbb{R}}$

Case 1: $\mathfrak{J}_{3\mathbb{R}}$ The $n = 2$ case $\mathfrak{J}_{1,1} = \mathbb{R} \oplus \Gamma_{1,1}$ may be written as $\mathfrak{J}_{3\mathbb{R}} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$. For $(a_1, a_2, a_3) \in \mathfrak{J}_{3\mathbb{R}}$ and $(a; a_\nu) \in \mathfrak{J}_{1,1}$ we have,

$$a_1 = a, \quad a_2 = a_0 + a_1, \quad a_3 = a_0 - a_1, \quad (2.45)$$

so that the cubic norm takes the more democratic form

$$N(A) = a_1 a_2 a_3. \quad (2.46)$$

While the analysis follows that of the generic $n > 2$ case, presented in Theorem 9, we highlight this form as it makes apparent the triality symmetry of the $n = 2$ cubic norm, which we will return to in section 3.4.3. There are just three irreducible idempotents:

$$E_1 = (1, 0, 0), \quad E_2 = (0, 1, 0), \quad E_3 = (0, 0, 1). \quad (2.47)$$

Case 2: $\mathfrak{J}_{2\mathbb{R}}$ For $n = 1$ the quadratic form on $\Gamma_{1, n-1}$ becomes Euclidean and Theorem 9 no longer holds. $\mathfrak{J}_{1,0}$ may be written as $\mathfrak{J}_{2\mathbb{R}} = \mathbb{R} \oplus \mathbb{R}$. For $A = (a, a_0) \in \mathfrak{J}_{2\mathbb{R}}$

$$N(A) = a(a_0)^2. \quad (2.48)$$

The symmetry of the cubic norm is reduced to $\text{SO}(1, 1)$ with a discrete factor, \mathbb{Z}_2 . Note, all rank 1 and 2 elements are respectively of the form $(a; 0)$ and $(0; a_0)$, where $a, a_0 \neq 0$. Consequently, unlike for $n > 1$, $\mathfrak{J}_{2\mathbb{R}}$ is *not* spanned by its rank 1 elements. There are just two irreducible idempotents:

$$E_1 = (1; 0), \quad E_2 = (-1; 1). \quad (2.49)$$

Case 3: $\mathfrak{J}_{\mathbb{R}}$ The sequence may be completed by defining $\mathfrak{J}_{\mathbb{R}} = \mathbb{R}$ with cubic norm,

$$N(A) = A^3, \quad A \in \mathbb{R}. \quad (2.50)$$

This cubic norm has no non-trivial symmetries. The unique choice of base point $c = 1$ yields $\text{Tr}(A) = 3A$, $\text{Tr}(A, B) = 3AB$, $A^\# = A^2$, from which it is clear that the cubic norm is Jordan. Note, all non-zero elements are rank 3 and there are no irreducible idempotents.

3 Orbits of Freudenthal Triple Systems

The Freudenthal triple system provides a natural representation of the dyonic black hole charge vectors for a broad class of 4-dimensional supergravity theories. In the following treatment, we present the axiomatic definition of the FTS which is manifestly covariant with respect to the 4-dimensional U-duality group G_4 . Subsequently, we present a particular realization in terms of Jordan algebras. This is equivalent to decomposing G_4 with respect to 5-dimensional U-duality group G_5 , which is modeled by the corresponding Jordan algebra. Consequently, this particular realization is manifestly covariant with respect to G_5 .

3.1 Axiomatic Definition of The FTS

An FTS is axiomatically defined [35] as a finite dimensional vector space \mathfrak{F} over a field \mathbb{F} (not of characteristic 2 or 3), such that:

1. \mathfrak{F} possesses a non-degenerate antisymmetric bilinear form $\{x, y\}$.
2. \mathfrak{F} possesses a symmetric four-linear form $q(x, y, z, w)$ which is not identically zero.
3. If the ternary product $T(x, y, z)$ is defined on \mathfrak{F} by $\{T(x, y, z), w\} = q(x, y, z, w)$, then

$$3\{T(x, x, y), T(y, y, y)\} = \{x, y\}q(x, y, y, y).$$

3.2 Definition Over a Cubic Jordan Algebra

Given a cubic Jordan algebra \mathfrak{J} defined over a field \mathbb{R} , there exists a corresponding FTS

$$\mathfrak{F}(\mathfrak{J}) = \mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{J} \oplus \mathfrak{J}. \quad (3.1)$$

An arbitrary element $x \in \mathfrak{F}(\mathfrak{J})$ may be written as a “ 2×2 matrix”

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}, \quad \text{where } \alpha, \beta \in \mathbb{R} \quad \text{and} \quad A, B \in \mathfrak{J}. \quad (3.2)$$

The FTS comes equipped with a non-degenerate bilinear antisymmetric quadratic form, a quartic form and a trilinear triple product:

1. Quadratic form $\{\bullet, \bullet\}: \mathfrak{F}(\mathfrak{J}) \times \mathfrak{F}(\mathfrak{J}) \rightarrow \mathbb{R}$

$$\{x, y\} := \alpha\delta - \beta\gamma + \text{Tr}(A, D) - \text{Tr}(B, C), \quad \text{where} \quad x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}, \quad y = \begin{pmatrix} \gamma & C \\ D & \delta \end{pmatrix}. \quad (3.3a)$$

2. Quartic form $\Delta: \mathfrak{F}(\mathfrak{J}) \rightarrow \mathbb{R}$

$$\Delta(x) := -(\alpha\beta - \text{Tr}(A, B))^2 - 4[\alpha N(A) + \beta N(B) - \text{Tr}(A^\#, B^\#)] =: \frac{1}{2}q(x). \quad (3.3b)$$

3. Triple product $T : \mathfrak{F}(\mathfrak{J}) \times \mathfrak{F}(\mathfrak{J}) \times \mathfrak{F}(\mathfrak{J}) \rightarrow \mathfrak{F}(\mathfrak{J})$ which is uniquely defined by

$$\{T(x, y, w), z\} = 2\Delta(x, y, w, z), \quad (3.3c)$$

where $\Delta(x, y, w, z)$ is the full linearization of $\Delta(x)$ normalized such that $\Delta(x, x, x, x) = \Delta(x)$.

Explicitly, the triple product is given by

$$T(x) = \begin{pmatrix} -\alpha^2\beta + \text{Tr}(A, B) - N(B) & -(\beta B^\sharp - B \times A^\sharp) + (\alpha\beta - \text{Tr}(A, B))A \\ (\alpha A^\sharp - A \times B^\sharp) - (\alpha\beta - \text{Tr}(A, B))B & \alpha\beta^2 - \text{Tr}(A, B) + N(A) \end{pmatrix}. \quad (3.4)$$

Note that *all* the necessary definitions, such as the cubic and trace bilinear forms, are inherited from the underlying Jordan algebra \mathfrak{J} .

Remark 10. For the Jordan algebras introduced in section 2, we will use the short hand notation:

$$\mathfrak{F}^{\mathbb{A}} := \mathfrak{F}(\mathfrak{J}_3^{\mathbb{A}}), \quad \mathfrak{F}^{2,n} := \mathfrak{F}(\mathfrak{J}_{1,n-1}), \quad \mathfrak{F}^{6,n} := \mathfrak{F}(\mathfrak{J}_{5,n-1}), \quad \mathfrak{F}_{3\mathbb{R}} := \mathfrak{F}(\mathfrak{J}_{3\mathbb{R}}), \quad \mathfrak{F}_{2\mathbb{R}} := \mathfrak{F}(\mathfrak{J}_{2\mathbb{R}}), \quad \mathfrak{F}_{\mathbb{R}} := \mathfrak{F}(\mathfrak{J}_{\mathbb{R}}).$$

3.3 The Automorphism Group

Definition 11 (The automorphism group $\text{Aut}(\mathfrak{F})$). *The automorphism group of an FTS is defined as the set of invertible \mathbb{R} -linear transformations preserving the quartic and quadratic forms:*

$$\text{Aut}(\mathfrak{F}) := \{\sigma \in \text{Iso}_{\mathbb{R}}(\mathfrak{F}) | \{\sigma x, \sigma y\} = \{x, y\}, \Delta(\sigma x) = \Delta(x)\}. \quad (3.5)$$

Note, the conditions $\{\sigma x, \sigma y\} = \{x, y\}$ and $\Delta(\sigma x) = \Delta(x)$ immediately imply

$$T(\sigma x) = \sigma T(x). \quad (3.6)$$

For $\mathfrak{F}^{\mathbb{A}(s)}$ the automorphism group has a two element centre and its quotient yields the simple groups listed in Table 1, while for $\mathfrak{F}^{2,n}$ one obtains the semi-simple groups $\text{SL}(2, \mathbb{R}) \times \text{SO}(2, n)$ [5, 34, 35]. In all cases \mathfrak{F} forms a symplectic representation of $\text{Aut}(\mathfrak{F})$, the dimensions of which are listed in the final column of Table 1. This table covers a number 4-dimensional supergravities: $\mathfrak{F}^{2,n}, \mathfrak{F}^{6,n} \rightarrow \mathcal{N} = 2, 4$ Maxwell-Einstein supergravity, $\mathfrak{F}^{\mathbb{A}} \rightarrow \mathcal{N} = 2$ “magic” Maxwell-Einstein supergravity and $\mathfrak{F}^{0^s} \rightarrow \mathcal{N} = 8$ maximally supersymmetric supergravity (see, for example, [6, 15, 24, 25, 36, 37]). Moreover, the special case of $\mathfrak{F}_{3\mathbb{R}}$ (and its generalisations) has found applications in the theory of entanglement [38–40].

Lemma 12. *The Lie algebra $\mathfrak{Aut}(\mathfrak{F})$ of $\text{Aut}(\mathfrak{F})$ is given by*

$$\mathfrak{Aut}(\mathfrak{F}) = \{\phi \in \text{Hom}_{\mathbb{R}}(\mathfrak{F}) | \Delta(\phi x, x, x, x) = 0, \{\phi x, y\} + \{x, \phi y\} = 0, \forall x, y \in \mathfrak{F}\}. \quad (3.7)$$

Proof. If $\phi \in \text{Hom}_{\mathbb{R}}(\mathfrak{F})$ satisfies $\Delta(e^{t\phi}x, e^{t\phi}x, e^{t\phi}x, e^{t\phi}x) = \Delta(x, x, x, x)$, where $t \in \mathbb{R}$, differentiating with respect to t and then setting $t = 0$ one obtains $\Delta(\phi x, x, x, x) = 0$. Similarly, if $\{e^{t\phi}x, e^{t\phi}y\} = \{x, y\}$, then $\{\phi x, y\} + \{x, \phi y\} = 0$. Conversely, assuming $\{\phi x, y\} + \{x, \phi y\} = 0$ for all $x, y \in \mathfrak{F}$ let $\sigma = e^{t\phi}$. Then,

$$\begin{aligned} \{e^{t\phi}x, e^{t\phi}y\} &= \{(1 + t\phi + \tfrac{1}{2}t^2\phi^2 + \dots)x, (1 + t\phi + \tfrac{1}{2}t^2\phi^2 + \dots)y\} \\ &= \{x, y\} + t(\{\phi x, y\} + \{x, \phi y\}) \\ &\quad + t^2(\tfrac{1}{2}\{\phi x, \phi y\} + \tfrac{1}{2}\{\phi^2x, y\} + \tfrac{1}{2}\{\phi x, \phi y\} + \tfrac{1}{2}\{x, \phi^2y\}) + \dots \\ &= \{x, y\}. \end{aligned} \quad (3.8)$$

Similarly, assuming $\Delta(\phi x, x, x, x) = 0$ and letting $\sigma = e^{t\phi}$, then $\Delta(\sigma x) = \Delta(x)$. \square

The automorphism group may also be defined in terms of the *Freudenthal product* [1, 33]. The Freudenthal product is useful in that it can be used to form elements of the Lie algebra and, further, we will need it to distinguish the orbits.

Table 1: The automorphism group $\text{Aut}(\mathfrak{F}(\mathfrak{J}))$ and the dimension of its representation $\dim \mathfrak{F}(\mathfrak{J})$ given by the Freudenthal construction defined over the cubic Jordan algebra \mathfrak{J} (with dimension $\dim \mathfrak{J}$ and reduced structure group $\text{Str}_0(\mathfrak{J})$).

Jordan algebra \mathfrak{J}	$\text{Str}_0(\mathfrak{J})$	$\dim \mathfrak{J}$	$\text{Aut}(\mathfrak{F}(\mathfrak{J}))$	$\dim \mathfrak{F}(\mathfrak{J})$
\mathbb{R}	—	1	$\text{SL}(2, \mathbb{R})$	4
$\mathbb{R} \oplus \mathbb{R}$	$\text{SO}(1, 1)$	2	$\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$	6
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	$\text{SO}(1, 1) \times \text{SO}(1, 1)$	3	$\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$	8
$\mathbb{R} \oplus \Gamma_{1, n-1}$	$\text{SO}(1, 1) \times \text{SO}(1, n-1)$	$n+1$	$\text{SL}(2, \mathbb{R}) \times \text{SO}(2, n)$	$2(n+2)$
$\mathbb{R} \oplus \Gamma_{5, n-1}$	$\text{SO}(1, 1) \times \text{SO}(5, n-1)$	$n+5$	$\text{SL}(2, \mathbb{R}) \times \text{SO}(6, n)$	$2(n+6)$
$\mathfrak{J}_3^{\mathbb{R}}$	$\text{SL}(3, \mathbb{R})$	6	$\text{Sp}(6, \mathbb{R})$	14
$\mathfrak{J}_3^{\mathbb{C}}$	$\text{SL}(3, \mathbb{C})$	9	$\text{SU}(3, 3)$	20
$\mathfrak{J}_3^{\mathbb{C}^s}$	$\text{SL}(3, \mathbb{R}) \times \text{SL}(3, \mathbb{R})$	9	$\text{SL}(6, \mathbb{R})$	20
$\mathfrak{J}_3^{\mathbb{H}}$	$\text{SU}^*(6)$	15	$\text{SO}^*(12)$	32
$\mathfrak{J}_3^{\mathbb{H}^s}$	$\text{SL}(6, \mathbb{R})$	15	$\text{SO}(6, 6)$	32
$\mathfrak{J}_3^{\mathbb{O}}$	$E_{6(-26)}$	27	$E_{7(-25)}$	56
$\mathfrak{J}_3^{\mathbb{O}^s}$	$E_{6(6)}$	27	$E_{7(7)}$	56

Definition 13 (Freudenthal product). For $x = (\alpha, \beta, A, B)$, $y = (\delta, \gamma, C, D)$, define the Freudenthal product

$$\wedge : \mathfrak{F} \times \mathfrak{F} \rightarrow \text{Hom}_{\mathbb{R}}(\mathfrak{F})$$

by,

$$x \wedge y := \Phi(\phi, X, Y, \nu), \quad \text{where} \quad \begin{cases} \phi &= -(A \vee D + B \vee C) \\ X &= -\frac{1}{2}(B \times D - \alpha C - \delta A) \\ Y &= \frac{1}{2}(A \times C - \beta D - \gamma B) \\ \nu &= \frac{1}{4}(\text{Tr}(A, D) + \text{Tr}(C, B) - 3(\alpha\gamma + \beta\delta)), \end{cases} \quad (3.9)$$

and $A \vee B \in \mathfrak{Str}_0(\mathfrak{J})$ is defined by $(A \vee B)C = \frac{1}{2} \text{Tr}(B, C)A + \frac{1}{6} \text{Tr}(A, B)C - \frac{1}{2}B \times (A \times C)$. The action of $\Phi : \mathfrak{F} \rightarrow \mathfrak{F}$ is given by

$$\Phi(\phi, X, Y, \nu) \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} = \begin{pmatrix} \alpha\nu + (Y, B) & \phi A - \frac{1}{3}\nu A + 2Y \times B + \beta X \\ -{}^t\phi B + \frac{1}{3}\nu B + 2X \times A + \alpha Y & -\beta\nu + (X, A) \end{pmatrix}. \quad (3.10)$$

Lemma 14. The automorphism group is given by the set of invertible \mathbb{R} -linear transformations preserving the Freudenthal product:

$$\text{Aut } \mathfrak{F} = \{\sigma \in \text{Iso}_{\mathbb{R}}(\mathfrak{F}) \mid \sigma(x \wedge y)\sigma^{-1} = \sigma x \wedge \sigma y\}. \quad (3.11)$$

Proof. We proceed by establishing the equivalence

$$\sigma(x \wedge y)\sigma^{-1} = \sigma x \wedge \sigma y \Leftrightarrow \{\sigma x, \sigma y\} = \{x, y\}, \quad \Delta(\sigma x) = \Delta(x). \quad (3.12)$$

We begin with the right implication \Rightarrow . First, following [33], we show that $\sigma(x \wedge y)\sigma^{-1} = \sigma x \wedge \sigma y \Rightarrow \{\sigma x, \sigma y\} = \{x, y\}$. Using the identity [33],

$$(x \wedge y)x - (x \wedge x)y + \frac{3}{8}\{x, y\}x = 0, \quad (3.13)$$

it follows that $(\sigma x \wedge \sigma y)\sigma x - (\sigma x \wedge \sigma x)\sigma y + \frac{3}{8}\{\sigma x, \sigma y\}\sigma x = 0$, which, from our assumption $\sigma(x \wedge y)\sigma^{-1} =$

$\sigma x \wedge \sigma y$ implies

$$\begin{aligned}
& \sigma(x \wedge y)x - \sigma(x \wedge x)y + \frac{3}{8}\{\sigma x, \sigma y\}\sigma x = 0 \\
& \Rightarrow \sigma(-\frac{3}{8}\{x, y\}x) + \frac{3}{8}\{\sigma x, \sigma y\}\sigma x = 0 \\
& \Rightarrow \frac{3}{8}(\{\sigma x, \sigma y\} - \{x, y\})\sigma x = 0 \\
& \Rightarrow \{\sigma x, \sigma y\} = \{x, y\}.
\end{aligned} \tag{3.14}$$

Since $\frac{2}{3}(x \wedge x)x = T(x)$, $\sigma(x \wedge y)\sigma^{-1} = \sigma x \wedge \sigma y$ implies $T(\sigma x) = \sigma T(x)$ and therefore $\{\sigma x, \sigma y\} = \{x, y\} \Rightarrow \Delta(\sigma x) = \Delta(x)$.

To establish the left implication \Leftarrow , we begin by noting that $T(x, y, z) = \frac{2}{9}[(x \wedge y)z + (y \wedge z)x + (z \wedge x)y]$. Then from the identity,

$$2(x \wedge y)z - (x \wedge z)y - (y \wedge z)x + \frac{3}{8}\{z, y\}x - \frac{3}{8}\{x, z\}y = 0, \tag{3.15}$$

which is easily obtained from (3.13), see Lemma 4.1.1 of [33], we have,

$$-\frac{9}{2}T(x, y, z) + 3(x \wedge y)z + \frac{3}{8}\{z, y\}x - \frac{3}{8}\{x, z\}y = 0. \tag{3.16}$$

Since $\sigma T(x, y, z) = T(\sigma x, \sigma y, \sigma z)$, our assumption $\{\sigma x, \sigma y\} = \{x, y\}$ together with (3.16) implies

$$\sigma[-\frac{9}{2}T(x, y, z) + \frac{3}{8}\{z, y\}x - \frac{3}{8}\{x, z\}y] + 3(\sigma x \wedge \sigma y)\sigma z = 0 \tag{3.17}$$

and so, substituting $-3(x \wedge y)\sigma^{-1}\sigma z$ in the square parentheses of (3.17),

$$3[(\sigma x \wedge \sigma y) - \sigma(x \wedge y)\sigma^{-1}]\sigma z = 0 \tag{3.18}$$

implying $(\sigma x \wedge \sigma y) = \sigma(x \wedge y)\sigma^{-1}$, as required. \square

Finally, we recall the explicit elements of $\text{Aut}(\mathfrak{F})$:

Lemma 15 (Seligman, 1962; Brown, 1969 [35, 41]). *The following transformations generate elements of $\text{Aut}(\mathfrak{F})$:*

$$\begin{aligned}
\varphi(C) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha + (B, C) + (A, C^\#) + \beta N(C) & A + \beta C \\ B + A \times C + \beta C^\# & \beta \end{pmatrix}; \\
\psi(D) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha & A + B \times D + \alpha D^\# \\ B + \alpha D & \beta + (A, D) + (B, D^\#) + \alpha N(D) \end{pmatrix}; \\
\hat{\tau} : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \lambda^{-1}\alpha & \tau A \\ {}^t\tau^{-1}B & \lambda\beta \end{pmatrix};
\end{aligned} \tag{3.19}$$

where $C, D \in \mathfrak{F}$ and $\tau \in \text{Str}(\mathfrak{F})$ s.t. $N(\tau A) = \lambda N(A)$.

For convenience we further define $\mathcal{Z} := \phi(-\mathbf{1})\psi(\mathbf{1})\phi(-\mathbf{1})$,

$$\mathcal{Z} : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} -\beta & -B \\ A & \alpha \end{pmatrix}. \tag{3.20}$$

3.3.1 Rank Conditions and $\text{Aut}(\mathfrak{F})$ -Equivalence

Lemma 16 (Krutelevich 2004). *Every non-zero element of $\mathfrak{F}(\mathfrak{J})$, where \mathfrak{J} is one of $\mathfrak{J}_3^{\mathbb{A}}, \mathfrak{J}_3^{\mathbb{A}^s}, \mathfrak{J}_{1,n-1}$ or $\mathfrak{J}_{3\mathbb{R}}, \mathfrak{J}_{2\mathbb{R}}, \mathfrak{J}_{\mathbb{R}}$, can be brought into the form*

$$x_{red} = \begin{pmatrix} 1 & A \\ 0 & \beta \end{pmatrix} \quad (3.21)$$

by $\text{Aut}(\mathfrak{F})$.

Proof. The proof presented by Krutelevich [5] for $\mathfrak{J}_3^{\mathbb{A}^s}$ also holds for $\mathfrak{J}_3^{\mathbb{A}}, \mathfrak{J}_{1,n-1}, \mathfrak{J}_{3\mathbb{R}}$, since it only requires that \mathfrak{J} is spanned by its rank 1 elements, and that the bilinear trace form is non-degenerate. However, $\mathfrak{J}_{2\mathbb{R}}$ and $\mathfrak{J}_{\mathbb{R}}$ are not spanned by their rank 1 elements² so a gentle modification of the proof is required. It is sufficient to show that the theorem holds for $\mathfrak{J}_{\mathbb{R}}$, so we focus on this case. Starting from an arbitrary element

$$\begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix},$$

we first show that one may always assume $\beta \neq 0$. Assume $\beta = 0$. If $\alpha \neq 0$ we simply apply $\phi(-1)\psi(1)\phi(-1)$. Now, assume $\alpha, \beta = 0$. This implies we may assume at least one of A or B are non-zero. Applying $\psi(D)$ we find

$$\beta = 0 \mapsto \beta' = 3D(BD + A)$$

so that we can always pick a D such that $\beta' \neq 0$. Hence, we may now assume from the outset that $\beta \neq 0$. We now proceed by illustrating that we may always assume $\alpha = 1$. Assume $\beta \neq 0$ and apply

$$\phi(C) : \alpha \mapsto \beta C^3 + 3AC^2 + 3BC + \alpha.$$

Since

$$\beta C^3 + 3AC^2 + 3BC + \alpha - 1 = 0$$

has at least one real root for $\beta \neq 0$ we have established that we may assume $\alpha = 1$. To finish the proof we assume $\alpha = 1$, and apply $\psi(-B)$. \square

Lemma 17. (i) *An element of $\mathfrak{F}(\mathfrak{J})$, where \mathfrak{J} is one of $\mathfrak{J}_3^{\mathbb{A}}, \mathfrak{J}_3^{\mathbb{A}^s}, \mathfrak{J}_{1,n-1}$, of the form*

$$\begin{pmatrix} \alpha & a_i E_i \\ 0 & \beta \end{pmatrix}, \quad i = 1, 2, 3, \quad (3.22)$$

is $\text{Aut}(\mathfrak{F})$ related to:

1.

$$\begin{pmatrix} \alpha & (a_1 + \beta c - \frac{a_2 a_3 c^2}{\alpha}) E_1 + a_2 E_2 + a_3 E_3 \\ 0 & \beta - \frac{2a_2 a_3 c}{\alpha} \end{pmatrix}. \quad (3.23)$$

2.

$$\begin{pmatrix} \alpha & a_1 E_1 + (a_2 + \beta c - \frac{a_1 a_3 c^2}{\alpha}) E_2 + a_3 E_3 \\ 0 & \beta - \frac{2a_1 a_3 c}{\alpha} \end{pmatrix}. \quad (3.24)$$

3.

$$\begin{pmatrix} \alpha & a_1 E_1 + a_2 E_2 + (a_3 + \beta c - \frac{a_1 a_2 c^2}{\alpha}) E_3 \\ 0 & \beta - \frac{2a_1 a_2 c}{\alpha} \end{pmatrix}. \quad (3.25)$$

²Indeed, $\mathfrak{J}_{\mathbb{R}}$ has no rank 1 elements.

(ii) An element of $\mathfrak{F}_{2\mathbb{R}}$, of the form

$$\begin{pmatrix} \alpha & a_i E_i \\ 0 & \beta \end{pmatrix} \quad i = 1, 2, \quad (3.26)$$

is $\text{Aut}(\mathfrak{F})$ related to:

$$\begin{pmatrix} \alpha & (a_1 + \beta c)E_1 + a_2 E_2 \\ 0 & \beta - \frac{2c(a_2)^2}{\alpha} \end{pmatrix}. \quad (3.27)$$

Proof. (i) The proof presented by Krutelevich [5] for $\mathfrak{J}_3^{\mathbb{A}}, \mathfrak{J}_3^{\mathbb{A}^s}$ may be seen to hold for $\mathfrak{J}_{1,n-1}$ by direct calculation. (ii) Follows by acting on (3.26) with

$$\psi(D) \circ \phi(C), \quad (3.28)$$

where $C = (c; 0)$ and $D = -\frac{1}{\alpha}(0; ca_2)$. \square

Following [5] one may generalise the conventional matrix rank to the FTS.

Definition 18 (FTS ranks). *An FTS element may be assigned an $\text{Aut}(\mathfrak{F})$ invariant rank:*

$$\begin{aligned} \text{Rank}x &= 1 \Leftrightarrow \Upsilon(x, x, y) = 0 \quad \forall y, \quad x \neq 0; \\ \text{Rank}x &= 2 \Leftrightarrow T(x) = 0, \quad \exists y \text{ s.t. } \Upsilon(x, x, y) \neq 0; \\ \text{Rank}x &= 3 \Leftrightarrow \Delta(x) = 0, \quad T(x) \neq 0; \\ \text{Rank}x &= 4 \Leftrightarrow \Delta(x) \neq 0, \end{aligned} \quad (3.29)$$

where we have defined $\Upsilon(x, x, y) := 3T(x, x, y) + \{x, y\}x$.

Remark 19 (Reduced rank conditions). *For an element in the reduced form (3.21) the rank conditions simplify:*

$$\begin{aligned} \text{Rank}x &= 1 \Leftrightarrow A = 0, \quad \beta = 0; \\ \text{Rank}x &= 2 \Leftrightarrow A^\sharp = 0, \quad \beta = 0, \quad A \neq 0; \\ \text{Rank}x &= 3 \Leftrightarrow 4N(A) = -\beta^2, \quad A^\sharp \neq 0; \\ \text{Rank}x &= 4 \Leftrightarrow 4N(A) \neq -\beta^2. \end{aligned} \quad (3.30)$$

In order to distinguish orbits of the same rank we will use the following quadratic form introduced in [1].

Definition 20 (FTS quadratic form). *Define, for a non-zero constant element $y \in \mathfrak{F}$, the real quadratic form B_y ,*

$$B_y(x) := \{(x \wedge x)y, y\}, \quad x \in \mathfrak{F}. \quad (3.31)$$

Lemma 21 (Shukuzawa, 2006 [1]). *If $y' = \sigma y$ for $y \neq 0$ and $\sigma \in \text{Aut}(\mathfrak{F})$, then*

$$B_y(x) = B_{y'}(x'), \quad \text{where } x' = \sigma x \in \mathfrak{F}.$$

3.4 Explicit Orbit Representatives

3.4.1 Magic FTS $\mathfrak{F}^{\mathbb{A}}$

Theorem 22 (Shukuzawa, 2006 [1]). *Every element $x \in \mathfrak{F}^{\mathbb{A}}$ of a given rank is $\text{Aut}(\mathfrak{F}^{\mathbb{A}})$ related to one of the following canonical forms:*

1. Rank 1

$$(a) \ x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

2. Rank 2

$$(a) \ x_{2a} = \begin{pmatrix} 1 & (1, 0, 0) \\ 0 & 0 \end{pmatrix}$$

$$(b) \ x_{2b} = \begin{pmatrix} 1 & (-1, 0, 0) \\ 0 & 0 \end{pmatrix}$$

3. Rank 3

$$(a) \ x_{3a} = \begin{pmatrix} 1 & (1, 1, 0) \\ 0 & 0 \end{pmatrix}$$

$$(b) \ x_{3b} = \begin{pmatrix} 1 & (-1, -1, 0) \\ 0 & 0 \end{pmatrix}$$

4. Rank 4

$$(a) \ x_{4a} = k \begin{pmatrix} 1 & (-1, -1, -1) \\ 0 & 0 \end{pmatrix}$$

$$(b) \ x_{4b} = k \begin{pmatrix} 1 & (1, 1, -1) \\ 0 & 0 \end{pmatrix}$$

$$(c) \ x_{4c} = k \begin{pmatrix} 1 & (1, 1, 1) \\ 0 & 0 \end{pmatrix},$$

where $k > 0$.

It is worth remarking here that the analogue of this Theorem for $\mathfrak{J}_3^{\text{O}^s}$ has been proved in [4] (see also [5]), and it holds for $\mathfrak{J}_3^{\text{H}^s}$ and $\mathfrak{J}_3^{\text{C}^s}$, as well.

3.4.2 Reducible Spin Factor FTS $\mathfrak{F}^{2,n}$

Theorem 23. *Every element $x \in \mathfrak{F}^{2,n}$ with $n \geq 2$ of a given rank is $\text{SL}(2, \mathbb{R}) \times \text{SO}(2, n)$ related to one of the following canonical forms:*

1. Rank 1

$$(a) \ x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

2. Rank 2

$$(a) \ x_{2a} = \begin{pmatrix} 1 & (1; 0, 0) \\ 0 & 0 \end{pmatrix}$$

$$(b) \ x_{2b} = \begin{pmatrix} 1 & (-1; 0, 0) \\ 0 & 0 \end{pmatrix}$$

$$(c) \ x_{2c} = \begin{pmatrix} 1 & (0; \frac{1}{2}, \frac{1}{2}) \\ 0 & 0 \end{pmatrix}$$

3. Rank 3

$$(a) \ x_{3a} = \begin{pmatrix} 1 & (0; 1, 0) \\ 0 & 0 \end{pmatrix}$$

$$(b) \ x_{3b} = \begin{pmatrix} 1 & (0; 0, 1) \\ 0 & 0 \end{pmatrix}$$

4. Rank 4

$$(a) \ x_{4a} = k \begin{pmatrix} 1 & (-1; 1, 0) \\ 0 & 0 \end{pmatrix}$$

$$(b) \ x_{4b} = k \begin{pmatrix} 1 & (1; 0, 1) \\ 0 & 0 \end{pmatrix}$$

$$(c) \ x_{4c} = k \begin{pmatrix} 1 & (-1; 0, 1) \\ 0 & 0 \end{pmatrix},$$

where $k > 0$.

Proof. We begin by transforming to the generic canonical form

$$\begin{pmatrix} 1 & A \\ 0 & \beta \end{pmatrix},$$

and then we proceed case by case, according to the rank.

Rank 1: $\text{Rank } x = 1 \Rightarrow A = 0, \beta = 0$, so that every rank 1 element is $\text{Aut}(\mathfrak{F}^{2,n})$ related to

$$x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.32)$$

Rank 2: $\text{Rank } x = 2 \Rightarrow A^\sharp = 0, \beta = 0, A \neq 0$, so that every rank 2 element is $\text{Aut}(\mathfrak{F}^{2,n})$ related to

$$x = \begin{pmatrix} 1 & A \\ 0 & 0 \end{pmatrix}, \quad (3.33)$$

where A is a rank 1 Jordan algebra element. A may be brought into canonical form via $\hat{\tau}$, where $\tau \in \text{Str}_0(\mathfrak{J}_{1,n-1})$,

$$\hat{\tau} : \begin{pmatrix} 1 & A \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \tau A \\ 0 & 0 \end{pmatrix}.$$

Hence, x may be brought into one of three forms corresponding to the three rank 1 representatives for A , namely:

$$x_{2a} = \begin{pmatrix} 1 & A_{1a} \\ 0 & 0 \end{pmatrix}; \quad x_{2b} = \begin{pmatrix} 1 & A_{1b} \\ 0 & 0 \end{pmatrix}; \quad x_{2c} = \begin{pmatrix} 1 & A_{1c} \\ 0 & 0 \end{pmatrix}. \quad (3.34)$$

These are in fact unrelated, as it can be seen by computing the quadratic forms

$$\begin{aligned} B_{x_{2a}}(y) &= -c_\mu c^\mu + \gamma d; \\ B_{x_{2b}}(y) &= c_\mu c^\mu - \gamma d; \\ B_{x_{2c}}(y) &= -cc_0 - cc_1 + \gamma d_0 + \gamma d_1, \end{aligned} \quad (3.35)$$

where

$$y = \begin{pmatrix} \delta & C \\ D & \gamma \end{pmatrix}. \quad (3.36)$$

By diagonalising (3.35), one can verify that the three forms have distinct signatures; hence, by Sylvester's Law of Inertia, x_{2a}, x_{2b} and x_{2c} lie in distinct orbits.

Rank 3: $\text{Rank} x = 3 \Rightarrow N(A) = -\frac{\beta^2}{4}, A^\# \neq 0$. If $\beta \neq 0$ then A is $\text{Str}_0(\mathfrak{J}_{1,n-1})$ related to

$$(\pm 1; \frac{1}{2}(1 \mp \frac{\beta^2}{4}), \frac{1}{2}(1 \pm \frac{\beta^2}{4}), 0, \dots) = \pm E_1 + E_2 \mp \frac{\beta^2}{4} E_3, \quad (3.37)$$

so that, by an application of $\hat{\tau}$, one obtains

$$x = \begin{pmatrix} 1 & \pm E_1 + E_2 \mp \frac{\beta^2}{4} E_3 \\ 0 & \beta \end{pmatrix}. \quad (3.38)$$

Then, by Lemma 17 with $c = \mp \frac{\beta^2}{4}$, x may be brought into the form,

$$\begin{pmatrix} 1 & \pm E_1 + E_2 \\ 0 & 0 \end{pmatrix}. \quad (3.39)$$

Hence, we may assume from the out set that

$$x = \begin{pmatrix} 1 & A \\ 0 & 0 \end{pmatrix} \quad (3.40)$$

where A is a rank 2 Jordan algebra element. Via an application of $\hat{\tau}$, where $\tau \in \text{Str}_0(\mathfrak{J}_{1,n-1})$, x may be brought into one of four forms corresponding to the four rank 2 canonical forms for A , namely:

$$x_{3a} = \begin{pmatrix} 1 & A_{2a} \\ 0 & 0 \end{pmatrix}; \quad x_{3b} = \begin{pmatrix} 1 & A_{2b} \\ 0 & 0 \end{pmatrix}; \quad x_{3c} = \begin{pmatrix} 1 & A_{2c} \\ 0 & 0 \end{pmatrix}; \quad x_{3d} = \begin{pmatrix} 1 & A_{2d} \\ 0 & 0 \end{pmatrix}. \quad (3.41)$$

We are now able to show x_{3a} and x_{3b} are $\text{Aut}(\mathfrak{J}^{2,n})$ related to x_{3d} and x_{3c} respectively. The proof proceeds by an application of $\varphi(\tilde{C})\psi(D)\varphi(C)$ with $\tilde{C}^\# = D^\# = C^\# = 0$, which yields,

$$\begin{pmatrix} 1 & A_{2a} \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 + (A_{2a} \times C + D, \tilde{C}) & A_{2a} + (A_{2a} \times C) \times D + (A_{2a}, D)\tilde{C} \\ A_{2a} \times C + D + (A_{2a} + (A_{2a} \times C) \times D) \times \tilde{C} & (A_{2a}, D) \end{pmatrix}.$$

Assuming $(A_{2a}, D) = 1$ and $\tilde{C} = -(A_{2a} + (A_{2a} \times C) \times D)$, one obtains

$$\begin{pmatrix} 1 - (A_{2a} \times C + D, A_{2a} + (A_{2a} \times C) \times D) & 0 \\ A_{2a} \times C + D & 1 \end{pmatrix}. \quad (3.42)$$

This is achieved by the choice $C = (0; -\frac{1}{2}, -\frac{1}{2}, 0, \dots)$ and $D = (0; \frac{1}{2}, \frac{1}{2}, 0, \dots)$. One finds $\tilde{C} = (0; -\frac{1}{2}, -\frac{1}{2}, 0, \dots)$ and $(A_{2a} \times C + D, \tilde{C}) = -1$, yielding

$$\begin{pmatrix} 0 & 0 \\ (-1; \frac{1}{2}, \frac{1}{2}, 0, \dots) & 1 \end{pmatrix},$$

from which, after three applications of $\varphi(-\mathbb{1})\psi(\mathbb{1})\varphi(-\mathbb{1})$, the desired form

$$\begin{pmatrix} 1 & A_{2d} \\ 0 & 0 \end{pmatrix}, \quad (3.43)$$

follows. Similarly, it can be proved that x_{2b} is $\text{Aut}(\mathfrak{J}^{2,n})$ related to x_{2c} .

The remaining two possibilities are unrelated, as it can be seen by computing the quadratic forms

$$\begin{aligned} B_{x_{3c}}(y) &= -\delta c_0 - \delta c_1 - cc_0 - cc_1 - c_\mu c^\mu + \gamma d + \gamma d_0 + \gamma d_1 + dd_0 + dd_1; \\ B_{x_{3d}}(y) &= \delta c_0 + \delta c_1 - cc_0 - cc_1 + c_\mu c^\mu - \gamma d + \gamma d_0 + \gamma d_1 - dd_0 - dd_1. \end{aligned} \quad (3.44)$$

By diagonalising (3.44), one can verify that the two forms have distinct signatures; hence, by Sylvester's Law of Inertia, x_{2a} and x_{2b} lie in distinct orbits.

Rank 4: $\text{Rank} x = 4 \Rightarrow \Delta(x) = -N(A) - \frac{\beta^2}{4} \neq 0$. By Lemma 17, we may assume from the out set that

$$x = \begin{pmatrix} 1 & A \\ 0 & 0 \end{pmatrix} \quad (3.45)$$

where A is a rank 3 Jordan algebra element. Via an application of $\hat{\tau}$, where $\tau \in \text{Str}_0(\mathfrak{J}_{1,n-1})$, x may be brought into one of two forms corresponding to the two rank 3 canonical forms for A , namely:

$$x_{4a} = \begin{pmatrix} 1 & (1; \frac{1}{2}(1-m), \frac{1}{2}(1+m), 0, \dots) \\ 0 & 0 \end{pmatrix}, \quad x_{4b} = \begin{pmatrix} 1 & (-1; \frac{1}{2}(1+m), \frac{1}{2}(1-m), 0, \dots) \\ 0 & 0 \end{pmatrix}, \quad (3.46)$$

where conventions have been chosen such that $\Delta(x_{4a}) = \Delta(x_{4b}) = m$.

In order to determine under what conditions x_{4a} and x_{4b} are related, use the quadratic forms

$$\begin{aligned} B_{x_{4a}}(y) &= \delta mc - \delta c_0 + \delta mc_0 - cc_0 + mcc_0 - \delta c_1 - \delta mc_1 - cc_1 - mcc_1 - c_\mu c^\mu \\ &\quad + \gamma d + \gamma d_0 - \gamma md_0 + dd_0 - mdd_0 - md_\mu d^\mu + \gamma d_1 + \gamma md_1 + dd_1 + mdd_1; \\ B_{x_{4b}}(y) &= -\delta mc + \delta c_0 + \delta mc_0 - cc_0 - mcc_0 + \delta c_1 - \delta mc_1 - cc_1 + mcc_1 - c_\mu c^\mu \\ &\quad - \gamma d + \gamma d_0 + \gamma md_0 - dd_0 - mdd_0 - md_\mu d^\mu + \gamma d_1 - \gamma md_1 - dd_1 + mdd_1. \end{aligned} \quad (3.47)$$

is made once again.

The diagonalisation of Eq. (3.47) leads to quite complicated expressions for the two metrics. However, one can show that they only differ in three components, namely $(1, -\frac{m}{2}, m)$ *versus* $(-1, -m, -\frac{m}{2})$; hence, one can conclude that for $m > 0$ the metrics have different signatures. Consequently, for $m > 0$, x_{4a} and x_{4b} lie in distinct orbits by Sylvester's Law of Inertia. On the other hand, for $m < 0$, the signatures match and, by using a similar argument to the one used in the rank 3 case, that is applying $\varphi(\tilde{C})\psi(D)\varphi(C)$ such that $\tilde{C}^\# = D^\# = C^\# = 0$, one can indeed verify they are both indeed related to the canonical form x_{4c} of the theorem. \square

Note, the $\mathfrak{F}^{2,n}$ case considered here may be generalised to an arbitrary pseudo-Euclidean signature $\mathfrak{F}^{p,q} := \mathfrak{F}(\mathfrak{J}_{p-1,q-1})$, where $\mathfrak{J}_{p-1,q-1} = \mathbb{R} \oplus \Gamma_{p-1,q-1}$ was introduced in (2.43). The automorphism group is given by,

$$\text{Aut}(\mathfrak{F}^{p,q}) = \text{SL}(2, \mathbb{R}) \times \text{SO}(p, q). \quad (3.48)$$

In particular, $\mathcal{N} = 4$ Maxwell-Einstein supergravity has an $\text{SL}(2, \mathbb{R}) \times \text{SO}(6, q)$ U-duality and is related to $\mathfrak{F}^{6,q} := \mathfrak{F}(\mathfrak{J}_{5,q-1})$. The analysis goes through analogously so we will not treat it in detail here. See, for example, [5, 6, 34] for further details.

3.4.3 Special Cases: $\mathfrak{F}_{3\mathbb{R}}$, $\mathfrak{F}_{2\mathbb{R}}$ and $\mathfrak{F}_{\mathbb{R}}$

Case 1: $\mathfrak{F}(\mathfrak{J}_{3\mathbb{R}})$ This is the $n = 2$ point of the generic sequence $\mathfrak{F}^{2,n}$, as presented in Theorem 23. However, as mentioned in section 2.3.3, the underlying Jordan algebra $\mathfrak{J}_{1,1} = \mathbb{R} \oplus \Gamma_{1,1}$ may be reformulated in particularly symmetric manner as $\mathfrak{J}_{3\mathbb{R}} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, where $N(A) = a_1 a_2 a_3$ for $(a_1, a_2, a_3) \in \mathfrak{J}_{3\mathbb{R}}$. The permutation symmetry of the cubic norm is further manifested in the corresponding FTS, $\mathfrak{F}^{2,2} \cong \mathfrak{F}_{3\mathbb{R}}$. The elements of $\mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{J}_{3\mathbb{R}} \oplus \mathfrak{J}_{3\mathbb{R}}$ may be written as a $2 \times 2 \times 2$ *hypermatrix*, denoted a_{ABC} :

$$x = \begin{pmatrix} a_{000} & A = (a_{011}, a_{101}, a_{110}) \\ B = (a_{100}, a_{010}, a_{001}) & a_{111} \end{pmatrix} \mapsto a_{ABC}, \quad \text{where } A, B, C = 0, 1. \quad (3.49)$$

The permutation symmetry of the cubic norm corresponds to its invariance under $A \leftrightarrow B \leftrightarrow C$. The hypermatrix lies in the fundamental representation $V_A \otimes V_B \otimes V_C$, where V_i is a 2-dimensional real vector space, of the automorphism group $\text{SL}(2, \mathbb{R}) \times \text{SO}(2, 2) \cong \text{SL}_A(2, \mathbb{R}) \times \text{SL}_B(2, \mathbb{R}) \times \text{SL}_C(2, \mathbb{R})$. Explicitly,

$$a_{ABC} \mapsto \tilde{a}_{ABC} = M_A^{A'} N_B^{B'} P_C^{C'} a_{A'B'C'}, \quad (3.50)$$

where M, N, P are 2×2 matrices with determinant 1. The quartic norm is given by Cayley's *hyperdeterminant* $\text{Det } a_{ABC}$ [42],

$$\Delta(x) = -\text{Det } a = \frac{1}{2} \epsilon^{A_1 A_2} \epsilon^{B_1 B_2} \epsilon^{C_1 C_3} \epsilon^{A_3 A_4} \epsilon^{B_3 B_4} \epsilon^{C_2 C_4} a_{A_1 B_1 C_1} a_{A_2 B_2 C_2} a_{A_3 B_3 C_3} a_{A_4 B_4 C_4}, \quad (3.51)$$

where ϵ is the antisymmetric 2×2 invariant tensor of $\text{SL}(2)$. This form of the quartic norm makes the $A \leftrightarrow B \leftrightarrow C$ triality invariance manifest.

Case 2: $\mathfrak{F}(\mathfrak{J}_{2\mathbb{R}})$ This is the $n = 1$ point of the generic sequence $\mathfrak{F}^{2,n}$. However, since the underlying Jordan algebra $\mathfrak{J}_{1,0} = \mathbb{R} \oplus \Gamma_{1,0}$ is Euclidean the orbits of Theorem 23 containing light-like $a_\mu \in \Gamma_{1,n-1}$ cannot be present. Indeed, we have the following result:

Theorem 24. *Every element $x \in \mathfrak{F}_{2\mathbb{R}}$ of a given rank is $\text{SL}(2, \mathbb{R}) \times \text{SO}(2, 1)$ related to one of the following canonical forms:*

1. Rank 1

$$(a) \ x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

2. Rank 2

$$(a) \ x_{2a} = \begin{pmatrix} 1 & (1; 0) \\ 0 & 0 \end{pmatrix}$$

$$(b) \ x_{2b} = \begin{pmatrix} 1 & (-1; 0) \\ 0 & 0 \end{pmatrix}$$

3. Rank 3

$$(a) \ x_{3a} = \begin{pmatrix} 1 & (0; 1) \\ 0 & 0 \end{pmatrix}$$

4. Rank 4

$$(a) \ x_{4a} = k \begin{pmatrix} 1 & (-1; 1) \\ 0 & 0 \end{pmatrix}$$

$$(b) \ x_{4b} = k \begin{pmatrix} 1 & (1; 1) \\ 0 & 0 \end{pmatrix},$$

where $k > 0$.

Proof. Since this is essentially a simplification of Theorem 23 we will not present the details here. The key observation is that, since every rank 1, 2 and 3 element of $\mathfrak{J}_{1,0}$ is respectively of the form $(a; 0)$, $(0; a_0)$ and $(a; a_0)$, where $a, a_0 \neq 0$, the x_{2c}, x_{3b} and x_{4c} orbits of Theorem 23 are absent. \square

The underlying Jordan algebra $\mathfrak{J}_{1,0} = \mathbb{R} \oplus \Gamma_{1,0}$ may be written as a degeneration $\mathfrak{J}_{3\mathbb{R}} \rightarrow \mathfrak{J}_{2\mathbb{R}} = \mathbb{R} \oplus \mathbb{R}$. At the level of the FTS $\mathfrak{F}_{3\mathbb{R}} \rightarrow \mathfrak{F}_{2\mathbb{R}}$, this corresponds to symmetrizing the $2 \times 2 \times 2$ hypermatrix of $\mathfrak{F}_{3\mathbb{R}}$ over two indices: $a_{ABC} \rightarrow a_{A(B_1 B_2)}$. The partially symmetrized hypermatrix lies in the $V_A \otimes \text{Sym}^2(V_B)$ representation of the automorphism group $\text{SL}(2, \mathbb{R}) \times \text{SO}(2, 1) \cong \text{SL}_A(2, \mathbb{R}) \times \text{SL}_B(2, \mathbb{R})$. Explicitly,

$$a_{A(B_1 B_2)} \mapsto \tilde{a}_{A(B_1 B_2)} = M_A^{A'} N_{B_1}^{B'_1} N_{B_2}^{B'_2} a_{A'(B'_1 B'_2)}, \quad (3.52)$$

where M, N are 2×2 matrices with determinant 1. The quartic norm is again given by the hyperdeterminant through applying the appropriate symmetrization to $\text{Det } a_{ABC}$. For more details see, for example, [5, 43, 44].

Case 3: $\mathfrak{F}(\mathfrak{J}_{\mathbb{R}})$ May be regarded as the end point of this sequence, in the sense that $\mathfrak{F}_{\mathbb{R}} = \mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{J}_{\mathbb{R}} \oplus \mathfrak{J}_{\mathbb{R}}$ can be mapped to the space of totally symmetrized hypermatrices: $a_{(A_1 A_2 A_3)} \in \text{Sym}^3(V_A)$. The totally symmetrized hypermatrix transforms as

$$a_{(A_1 A_2 A_3)} \mapsto \tilde{a}_{(A_1 A_2 A_3)} = M_{A_1}^{A'_1} M_{A_2}^{A'_2} M_{A_3}^{A'_3} a_{(A'_1 A'_2 A'_3)}, \quad (3.53)$$

where M is a 2×2 determinant 1 matrix, under the automorphism group $\text{SL}_A(2, \mathbb{R})$. Once again the quartic norm is given by the hyperdeterminant by totally “symmetrizing” $\text{Det } a_{ABC}$. For more details see, for example, [5, 43, 44].

As already noted in section 2.3.3, because $N(A) = A^3, A^\sharp = A^2$, all non-zero elements $A \in \mathfrak{J}_{\mathbb{R}}$ are rank 3. Consequently the number of independent ranks in $\mathfrak{F}_{\mathbb{R}}$ is reduced to three:

Lemma 25. *If $x \in \mathfrak{F}_{\mathbb{R}}$ is rank 2, then it is rank 1.*

Proof. Consider the independent rank 2 conditions evaluated on the reduced form of (3.21):

$$\text{Rank } x_{\text{red}} \leq 2 \Leftrightarrow A^\sharp = 0, \beta = 0. \quad (3.54)$$

Since $A^\sharp = 0 \Rightarrow A = 0$ for all $A \in \mathfrak{J}_{\mathbb{R}}$, one obtains

$$\text{Rank } x_{\text{red}} \leq 2 \Rightarrow A = 0, \beta = 0 \Rightarrow \text{Rank } x_{\text{red}} = 1. \quad (3.55)$$

□

Hence, the rank 2 orbits of Theorem 23 do not exist for $\mathfrak{F}_{\mathbb{R}}$. There are only elements of rank 1, 3 or 4, and we have the following

Theorem 26. *Every element $x \in \mathfrak{F}_{\mathbb{R}}$ of a given rank is $\text{SL}(2, \mathbb{R})$ related to one of the following canonical forms:*

1. Rank 1

$$(a) \ x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

2. Rank 3

$$(a) \ x_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

3. Rank 4

$$(a) \ x_{4a} = k \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$(b) \ x_{4b} = k \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

where $k > 0$.

Proof. We begin by transforming to the generic canonical form

$$\begin{pmatrix} 1 & A \\ 0 & \beta \end{pmatrix}, \quad (3.56)$$

and proceed, case by case, according to the rank.

Rank 1: $\text{Rank} x = 1 \Rightarrow A = 0, \beta = 0$, so that every rank 1 element is $\text{Aut}(\mathfrak{F}_{\mathbb{R}})$ related to

$$x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.57)$$

Rank 3: $\text{Rank} x = 3 \Rightarrow 4N(A) = 4A^3 = -\beta^2$ and $A \neq 0$, so that every rank 3 element is $\text{Aut}(\mathfrak{F}_{\mathbb{R}})$ related to a reduced form

$$\begin{pmatrix} 1 & A \\ 0 & \sqrt{-A^3} \end{pmatrix}, \quad (3.58)$$

with $A < 0$. In order to determine the $\text{Aut}(\mathfrak{F}_{\mathbb{R}})$ transformation bringing (3.58) into the desired form, it is convenient to use the totally symmetric hypermatrix representation of the charges:

$$x = a_{(A_1 A_2 A_3)}, \quad A_1, A_2, A_3 = 0, 1 \quad (3.59)$$

where, explicitly,

$$\begin{aligned} a_{(000)} &= \alpha, & a_{(110)} &= a_{(101)} = a_{(011)} = A; \\ a_{(000)} &= \beta, & a_{(001)} &= a_{(010)} = a_{(100)} = B. \end{aligned} \quad (3.60)$$

A generic $\text{Aut}(\mathfrak{F}_{\mathbb{R}})$ transformation is then given by an $\text{SL}(2, \mathbb{R})$ matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1 \quad (3.61)$$

under which,

$$a_{(ABC)} \mapsto M_A^{A'} M_B^{B'} M_C^{C'} a_{(A'B'C')} = \tilde{a}_{(ABC)}. \quad (3.62)$$

Applying M to the reduced form (3.58), we see that in order to obtain x_3 we are required to solve the follow system of polynomial equations

$$\tilde{a}_{(000)} = a^3 - 3|A|b^2a + 2|A|^{\frac{3}{2}}b^3 = 0; \quad (3.63)$$

$$\tilde{a}_{(110)} = c^2a - |A|(ad^2 + 2bcd) + 2|A|^{\frac{3}{2}}d^2b = \tilde{A}; \quad (3.64)$$

$$\tilde{a}_{(001)} = a^2c - |A|(cb^2 + 2dab) + 2|A|^{\frac{3}{2}}b^2d = 0; \quad (3.65)$$

$$\tilde{a}_{(111)} = c^3 - 3|A|d^2c + 2|A|^{\frac{3}{2}}d^3 = 0, \quad (3.66)$$

where we leave $\tilde{A} \neq 0$ arbitrary as it may be subsequently scaled away using

$$M = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A}^{-1} \end{pmatrix}. \quad (3.67)$$

Setting $d = 1$, one immediately sees that (3.66) has two distinct real roots:

$$(c + 2|A|^{\frac{1}{2}})(c - |A|^{\frac{1}{2}})^2 = 0. \quad (3.68)$$

The double root $c = |A|^{\frac{1}{2}}$ contradicts $\tilde{A} \neq 0$ so we choose $c = -2|A|^{\frac{1}{2}}$, which implies $a + 2|A|^{\frac{1}{2}}b = 1$ and, from (3.64):

$$\tilde{A} = 3|A|. \quad (3.69)$$

Substituting $d = 1, c = -2|A|^{\frac{1}{2}}$ into (3.65) and solving for a we find,

$$a_{\pm} = \frac{|A|^{\frac{1}{2}}b}{2}(-1 \pm 3). \quad (3.70)$$

Letting $a = a_+$ we determine that

$$M = \begin{pmatrix} \frac{1}{3} & \frac{1}{3|A|^{\frac{1}{2}}} \\ -2|A|^{\frac{1}{2}} & 1 \end{pmatrix} \quad (3.71)$$

sends (3.58) to $\begin{pmatrix} 0 & 3|A| \\ 0 & 0 \end{pmatrix}$, which is related by $M = \begin{pmatrix} 3|A| & 0 \\ 0 & (3|A|)^{-1} \end{pmatrix}$ to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, as required.

Rank 4: $\text{Rank} x = 4 \Rightarrow 4A^3 + \beta^2 \neq 0$. First, we show that starting from (3.56) every rank 4 x may be brought into a form with $B = \beta = 0$, namely:

$$\begin{pmatrix} \tilde{\alpha} & \tilde{A} \\ 0 & 0 \end{pmatrix}. \quad (3.72)$$

This amounts to solving

$$\tilde{a}_{(001)} = a^2c + A(cb^2 + 2dab) + \beta b^2d = 0; \quad (3.73)$$

$$\tilde{a}_{(111)} = c^3 + 3Ad^2c + \beta d^3 = 0, \quad (3.74)$$

where $ad - bc = 1$. There are three subcases: (i) $A \neq 0, \beta = 0$, (ii) $A = 0, \beta \neq 0$, (iii) $A \neq 0, \beta \neq 0$. (i) is trivial. For (ii), our system simplifies down to

$$\tilde{a}_{(001)} = a^2c + \beta b^2d = 0; \quad (3.75)$$

$$\tilde{a}_{(111)} = c^3 + \beta d^3 = 0, \quad (3.76)$$

Setting $d = 1$ and $c = -\beta^{\frac{1}{3}}$ solves (3.76). By substituting this choice into (3.75) and solving for a , one finds $a_{\pm} = \pm\beta^{\frac{1}{3}}b$. But only a_+ is consistent with $ad - bc = 1$. Making this choice implies $b = (8\beta)^{-\frac{1}{3}}$, and one obtains the $\text{SL}(2, \mathbb{R})$ matrix

$$M = \begin{pmatrix} \frac{1}{2} & -\beta^{\frac{1}{3}} \\ (8\beta)^{-\frac{1}{3}} & 1 \end{pmatrix}. \quad (3.77)$$

Finally, let us consider the case (iii) $A \neq 0, \beta \neq 0$. Let $c = \gamma d$, where $\gamma = \gamma(\beta, A)$. Then, from (3.74), we have

$$d^3(\gamma^3 + 3A\gamma + \beta) = 0, \quad (3.78)$$

which, since d is necessarily non-zero, implies

$$f(\gamma) = \gamma^3 + 3A\gamma + \beta = 0. \quad (3.79)$$

There is at least one real root γ_* that is non-zero for $\beta \neq 0$. Substituting into (3.73) yields,

$$\gamma_*d \left[a^2 + \frac{2Ab}{\gamma_*}a + \left(A + \frac{\beta}{\gamma_*}\right)b^2 \right] = 0. \quad (3.80)$$

Solving for a we find

$$a_{\pm} = \frac{Ab}{\gamma_*} \left(-1 \pm \sqrt{1 - \frac{\gamma_*^2}{A^2} \left(A + \frac{\beta}{\gamma_*}\right)} \right) = \xi_{\pm}(\beta, A)b. \quad (3.81)$$

Hence, we require

$$A^2 - \gamma_*^2 A - \gamma_* \beta = A^2 - y(\gamma_*) \geq 0, \quad (3.82)$$

where $y(\gamma_*) = \gamma_*^2 A + \gamma_* \beta$. We may always choose the root γ_* such that this condition holds. In order to see this, let us consider the two subcases: (a) $A < 0$ and (b) $A > 0$. (a) For $A < 0$, $f(\gamma)$ in (3.79) has two turning points at $\pm\sqrt{|A|}$. Consequently, for $\beta > 0$ there is always a real root $\gamma_* < -\sqrt{|A|} < 0$, which implies (3.82). Similarly, for $\beta < 0$ there is always a real root $\gamma_* > \sqrt{|A|} > 0$, which again implies (3.82). (b) For $A > 0$ the cubic $f(\gamma)$ only has an inflection point at $\gamma = 0$, $f(0) = \beta$ and so $f(\gamma) = 0$ has a single real root γ_* . If $\beta < 0$, then $0 < \gamma_* < \frac{|\beta|}{3A}$. Since $y(\gamma_*)$ has a minimum at $\frac{-\beta}{2A}$ such that $y(\frac{-\beta}{2A}) = \frac{-\beta^2}{4A} < 0$, it is clear that $0 < \gamma_* < \frac{|\beta|}{3A}$ implies $y(\gamma_*) < 0$ and so condition (3.82) is satisfied. Similarly, if $\beta > 0$, then $-\frac{\beta}{3A} < \gamma_* < 0$, and once again $y(\gamma_*) < 0$, implying condition (3.82) as required. Hence, we conclude condition (3.82) may always be satisfied.

In summary:

$$a = \xi_{\pm}(\beta, A)b, \quad c = \gamma_*(\beta, A)d \quad (3.83)$$

which yields

$$(\xi_{\pm} - \gamma_*)bd = 1, \quad (3.84)$$

where ξ_+ or ξ_- is chosen such that $\xi_{\pm} - \gamma_*$ is non-zero³. Without loss of generality we can set $d = 1$, so that $b = 1/(\xi_{\pm} - \gamma_*)$ and the $\text{SL}(2, \mathbb{R})$ matrix

$$M = \begin{pmatrix} \xi_{\pm}/(\xi_{\pm} - \gamma_*) & 1/(\xi_{\pm} - \gamma_*) \\ \gamma_* & 1 \end{pmatrix}, \quad (3.85)$$

transforms our hypermatrix into the desired form (3.72).

Finally, the reduced form (3.72) may be brought into the form $x_{4a/b}$ of the theorem by the diagonal $\text{SL}(2, \mathbb{R})$ transformation

$$M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \begin{pmatrix} \tilde{\alpha} & \tilde{A} \\ 0 & 0 \end{pmatrix} \mapsto k \begin{pmatrix} 1 & \epsilon \\ 0 & 0 \end{pmatrix}. \quad (3.86)$$

where $\epsilon = +1, -1$ according as $\Delta > 0, \Delta < 0$. □

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³Note that $\xi_{\pm} = \gamma_*$ for both choices implies $\gamma_* = \pm\sqrt{-A}$ which, from (3.79), implies $4A^3 + \beta^2 = 0$, in turn contradicting our rank 4 assumption.

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